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Localisation and Completions of Noetherian PI Algebras

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We present several examples of Noetherian rings that answer in the negative some well-known questions concerning localisation, completions and the principal ideal theorem. The main example is a prime, Noetherian, PI ring R with a maximal ideal P such that $\bigcap P^n = 0$, P is neither left nor right localisable and the completion of R at P is Noetherian. The only other maximal ideal of R is idempotent. The second ring contains a maximal ideal P, again with $\bigcap P^n = 0$, but such that P is left but not right localisable. The final example shows that the natural generalisation of the principal ideal theorem for Noetherian, PI rings is false.

The rings that we construct each depend upon the following example of Nagata (see, for example, [7, p. 327] for an exposition). There exists a pair of commutative Noetherian domains, $A \subset B$, such that:

(i) A is a local ring with maximal ideal M;

(ii) B is a finite A-module, such that B has exactly two maximal ideals, M_1 and M_2, and $M_1 \cap M_2 = M$. Note that this implies that M is an ideal of B;

(iii) M_1 has height 1, whereas M_2 has height 2.

Suppose that P is a maximal ideal of a Noetherian ring R, with $\bigcap P^n = 0$. A long outstanding question, due originally to Goldie [3, p. 101], asks whether P must be localisable. A related question, due to McConnell, asks whether P must be localisable for the completion of R at P to be Noetherian

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(in fact, the latter question was posed in the more general setting of a prime ideal P). The first example answers both questions in the negative.

EXAMPLE 1. There exists a prime ring R, finitely generated as a module over its Noetherian centre, such that R has exactly two maximal ideals, say P and Q. Furthermore:

(i) $Q^2 = Q$,

(ii) $\bigcap P^n = 0$,

(iii) P is neither left nor right localisable,

(iv) the completion of R at P is Noetherian.

Remark. Observe that, by [6], $\{P, Q\}$ must form a clan. Thus the prime ideals in a clan can have very different properties.

Proof. Before proceeding to the construction of R, we observe that, in Nagata's example, B/A is a simple A-module (in fact, even if this were not the case, this additional condition could be obtained by replacing B by $A + V$ for any $M \subset V \subseteq M_1$ with V/M simple as an A-module).

The ring that we are interested in is

$$R = \left\{ \begin{pmatrix} r+a & b \\ c & r \end{pmatrix} : r \in A, a, b \in M_1 \quad \text{and} \quad c \in M \right\},$$

which we shall write as

$$R = \begin{pmatrix} M_1 + A & M_1 \\ M & A \end{pmatrix}.$$

(Of course, $M_1 + A = B$, but we write it this way to emphasize the identification.)

The centre of R is $\{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : a \in A\}$, over which R is finite. R is a prime ring since it contains the ideal $\left(\begin{smallmatrix} M & M \\ M & M \end{smallmatrix}\right)$ of A_2, the 2×2 matrices over A. Observe that the ideal $J = \left(\begin{smallmatrix} M & M_1 \\ M & M \end{smallmatrix}\right)$ of R is quasiregular (use determinants!) and that $R/J \cong B/M$ is a semisimple ring. Thus $J = J(R)$, the Jacobson radical of R. Finally, set

$$P = \begin{pmatrix} M_1 & M_1 \\ M & M \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} M & M_1 \\ M & M \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} R,$$

where a is any element of M_2 for which $a - 1 \in M_1$. Then P and Q are distinct, proper ideals of R. Since R/J has length 2, $\{P, Q\}$ must be precisely the set of maximal ideals of R, and $P \cap Q = J(R)$.

(i) We start by calculating some subsets of Q^2. Certainly:

$$Q^2 \supseteq \begin{pmatrix} M & M_1 \\ M & M \end{pmatrix}^2 = \begin{pmatrix} MM_1 & MM_1 \\ M & MM_1 \end{pmatrix} \tag{1}$$

and

$$Q^2 \supseteq \begin{pmatrix} M & M_1 \\ M & M \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} M & M_1 \\ M & M \end{pmatrix} = \begin{pmatrix} aM & M_1 \\ M & M \end{pmatrix}. \tag{2}$$

Now, $M_1 = aA = B$ and so $aM + MM_1 = M$. Thus Eqs. 1 and 2 imply that

$$Q^2 \supseteq \begin{pmatrix} M & M_1 \\ M & M \end{pmatrix} = J(R).$$

Finally, $\left(\begin{smallmatrix} a^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in Q^2 \backslash J(R)$. Since $R/J(R)$ has length 2, this implies that $Q^2 = Q$.

(ii) Notice that

$$P \subset \begin{pmatrix} M_1 & M_1 \\ M_1 & M_1 \end{pmatrix}$$

which is an ideal of B_2, the 2×2 matrices over B. Since B is a commutative, Noetherian domain, $\bigcap (M_1)^n = 0$ and so $\bigcap P^n = 0$.

(iii) It follows from (i) and [6, Theorem 5] that P is not localisable. However, we can also give an easy, direct proof that P is neither left nor right localisable.

Let $a \in M_2$ be such that $a - 1 \in M_1$. Then $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathscr{C}(P)$, the elements regular mod P. Let $m \in M_1 \backslash aM_1$, which exists since $\bigcap a^n B = 0$. If $\mathscr{C}(P)$ satisfies the right Ore condition, then

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathscr{C}(P).$$

This forces b_4 to be a unit in A, whence $m = a(a_2 b_4^{-1}) \in aM_1$; a contradiction. A similar argument—essentially the transpose of the present one—shows that P is not left localisable.

(iv) Let $J = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : a \in M\}$, the maximal ideal of the centre of R. Let $I = P^2 + JR$.

Then

$$I^2 + JR \supseteq P^4 + JR = \begin{pmatrix} M_1^4 & M_1^4 \\ M_1^3 M & M_1^3 M \end{pmatrix} + \begin{pmatrix} MM_1 & MM_1 \\ M^2 & M^2 \end{pmatrix}. \tag{3}$$

Now, as M_1/M is a simple B-module, $M_1 = M + M_1^n$ for any $n \geqslant 1$. Combined with Eq. 3 this implies that

$$I^2 + JR \supseteq \begin{pmatrix} M_1^2 & M_1^2 \\ M_1 M & M_1 M \end{pmatrix} + JR = P^2 + JR = I.$$

Thus, in the notation of [5], I is polycentral mod I^2. So, by [5, Theorem 3.4], the completion of R at I is Noetherian. Finally, since $P^2 \subseteq I \subseteq P$, this completion equals the completion of R at P.

In [4], McConnell gives an example of a prime ideal P in a Noetherian domain R such that $\bigcap P^n = 0$ but P is not localisable and the completion of R at P is not Noetherian. It is fairly easy to obtain a ring that is a finite module over its centre with these properties. However, in these examples the intersection of the symbolic powers, $\bigcap P^{(n)} \neq 0$. (Observe that this implies that in any localisation of R in which P becomes a maximal ideal, $\bigcap P^n \neq 0$.) Thus, given McConnell's example and Example 1: Is it possible that the completion of a Noetherian ring R at a prime ideal P is Noetherian if and only if $\bigcap P^{(n)} = 0$?

It is easy to find a prime ideal P in a Noetherian ring R such that P is left but not right localisable. However, the standard examples of this only work because the left and right torsion radicals with respect to $\mathbb{C}(P)$ are not equal. The next example has a maximal ideal P, with $\bigcap P^n = 0$, such that P is left but not right localisable. Of course, in such an example both torsion radicals must be zero. Since the ring has symmetric properties, it illustrates the difficulties one encounters in trying to find localisation criteria.

EXAMPLE 2. There exists a ring R_1, finite as a module over its Noetherian centre, and a maximal ideal P_1 of R such that (i) $\bigcap P_1^n = 0$, and (ii) P_1 is left but not right localisable.

Proof. R_1 is the subring

$$\begin{pmatrix} M_1 + A \diagdown & M_1 \\ 0 & \diagdown A \end{pmatrix} = \left\{ \begin{pmatrix} a+b & c \\ 0 & a \end{pmatrix} : a \in R \quad \text{and} \quad b, c \in M_1 \right\}$$

of Example 1 and P_1 is the ideal $\left(\begin{smallmatrix} M_1 & M_1 \\ 0 & M \end{smallmatrix}\right)$. The proof used in Example 1 (iii) shows that P_1 is not right localisable.

It remains to show that P_1 is left localisable. Let $c = \left(\begin{smallmatrix} a & b \\ 0 & u \end{smallmatrix}\right) \in \mathscr{C}(P_1)$ and $r = \left(\begin{smallmatrix} f & g \\ 0 & h \end{smallmatrix}\right) \in R_1$. Since B is commutative and u is a unit in A,

$$\begin{pmatrix} a & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} f & g \\ 0 & h \end{pmatrix} = \begin{pmatrix} f & g' \\ 0 & h \end{pmatrix} \begin{pmatrix} a & b \\ 0 & u \end{pmatrix},$$

where $g' = (ag - fb)\, u^{-1}$. So P is left localisable.

It would be interesting to know if ther exists a ring R_1, satisfying the properties of Example 2, but such that R_1 is a prime ring.

Let R be a Noetherian PI algebra, $a \in R$ and T be the *bound* of aR; i.e., the largest two-sided ideal contained in aR. One would like to prove the following generalization of the principal ideal theorem: any prime ideal P minimal over T is of height 1. This has been achieved in [2, Theorem 4.8] if one makes the additional assumption that $a \notin \mathscr{C}(P)$. In our final example we show that this extra condition is necessary.

EXAMPLE 3. There exists a prime ring R, finite as a module over its Noetherian centre, with the following property: there exists $a \in R$ and P a maximal ideal, minimal over the bound of aR, such that P is of height 2.

Proof. Let $R = \begin{pmatrix} B & M \\ M & A \end{pmatrix}$, where $A \subset B$ are, again, the Nagata example. As before, R is a prime ring. Let $a \in M_1 \backslash M_2$. Since M_1 is the only prime ideal not contained in M_2, it must also be the only prime ideal that contains a. So $aB \supset M_1^n$ for some integer n. So, if $\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ then

$$\alpha R = \begin{pmatrix} aB & aM \\ M & A \end{pmatrix} \supset T = \begin{pmatrix} M_1^n & M_1^n M \\ M_1^n M & M_1^n M \end{pmatrix},$$

which is an ideal of R. Let $P = \begin{pmatrix} B & M \\ M & M \end{pmatrix}$ and $Q = \begin{pmatrix} M_1 & M \\ M & A \end{pmatrix}$. Then $(P \cap Q)^{n+1} \subset T$. However, since

$$Q^m = \begin{pmatrix} M_1^m + M^2 & M \\ M & A \end{pmatrix}$$

and $aM \neq M$, there is no integer m such that $Q^m \subset \alpha R$. Similarly, $P^m \not\subset \alpha R$ for every m. Thus both P and Q are minimal over the bound of αR.

Finally, we claim that P is a height 2 prime ideal. For, as M_2 has height 2, there exists a prime ideal $U \subset M_2$ such that U has height 1. Now, $U \cap A = V$ is a prime ideal of A that must also have height 1. Since A is local, this implies that $U \cap M = V$ and $SV \subseteq U \cap SM = V$. Thus $P_1 = \begin{pmatrix} U & V \\ V & V \end{pmatrix}$ is an ideal of R. Indeed, either by direct calculation or [1, Proposition 4], P_1 is actually a prime ideal. Since R is a prime ring, this implies that P is a height 2 prime ideal.

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